$L^1(\Omega, \mathcal{F}, \mathbb{P})$ is the linear space of (equivalence classes of) $\mathcal{F}$-measurable random variables, f, for which $\int_\Omega |f| \, d\mathbb{P} < \infty$. With,

$$\|f\|_1 = \int_\Omega |f| \, d\mathbb{P}$$

is a complete normed vector space. Every $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ can be written as $f = f^+ - f^-$ where each of f^+, f^- lie in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $f^+ \geq 0 \leq f^-$ with $f^+ f^- = 0$. So let f lie in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and suppose $f \geq 0$. Consider $f_n = f I_{\{f \leq n\}}$, then $f_n \leq n$ ($\mathbb{P}$-a.s) and hence $f_n \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subseteq L^2(\Omega, \mathcal{F}, \mathbb{P})$. Moreover $f_n \leq f_{n+1}$ ($\mathbb{P}$-a.s.) and $f_n \uparrow f$ ($\mathbb{P}$ a.s). The monotone convergence theorem tells us that $f_n \to f$ in the norm of L^1. In view of the fact that an arbitrary $g \in L^1$ can be written $g = g^+ - g^-$ (as above) then we conclude that g is the $\|\cdot\|_1$ limit of a sequence of elements from $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Indeed, we can choose this sequence to consist of bounded functions. Now Hölders Inequality shows that $L^2(\Omega, \mathcal{F}, \mathbb{P}) \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$ combining this with our remarks above we see that $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a norm dense subspace of $L^1(\Omega, \mathcal{F}, \mathbb{P})$.

We have already considered the conditional expectation of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ onto $L^2(\Omega, \mathcal{G}, \mathbb{P})$ when $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$. A moments thought shows that $L^1(\Omega, \mathcal{F}, \mathbb{P})$ will contain $L^1(\Omega, \mathcal{G}, \mathbb{P})$ as a closed subspace — the argument is identical to that used to establish $L^2(\Omega, \mathcal{G}, \mathbb{P})$ closed in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Can we extend the conditional expectation, $M_\mathcal{G} : L^2(\Omega, \mathcal{F}, \mathbb{P}) \xrightarrow{\text{onto}} L^2(\Omega, \mathcal{G}, \mathbb{P})$ to a map of $L^1(\Omega, \mathcal{F}, \mathbb{P})$ onto $L^1(\Omega, \mathcal{G}, \mathbb{P})$?

The answer is yes! First we need a result about the L^1-norm of elements of L^2.

<u>Lemma</u>

Let $f \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and G be a sub-σ-field of $\mathcal{F}$. Then $\| M_G(f) \|_1 \leqslant \| f \|_1$.

<u>Proof</u>

Write $f = f^+ - f^-$ and $g = I_{\{f \geqslant 0\}} - I_{\{f < 0\}}$

then $\| g \|_\infty = 1$ and $fg = f^+ + f^- = |f|$

so $\mathbb{E}(fg) = \mathbb{E}(|f|) = \| f \|_1$ and

$\| f \|_1 \leq \sup\limits_{\|g\|_\infty \leq 1} |\mathbb{E}(fg)|$. On the other hand

if $\| g \|_\infty \leqslant 1$ then by Jensens inequality

$$|\mathbb{E}(fg)| \leq \mathbb{E}(|fg|) = \mathbb{E}(|f| \cdot |g|) \leq \mathbb{E}(|f|)$$

since $\|g\|_\infty \leqslant 1 \Rightarrow |g| \leq 1$ $\mathbb{P}$ a.s. So

$$\sup_{\|g\|_\infty \leqslant 1} |\mathbb{E}(fg)| \leq \mathbb{E}(|f|).$$

We have proved,

$$\| f \|_1 = \mathbb{E}(|f|) = \sup_{\|g\|_\infty \leqslant 1} |\mathbb{E}(fg)|$$

Now we already know that $\| M_G(g) \|_\infty \leqslant \| g \|_\infty$

So, $\| M_G(f) \|_1 = \sup\limits_{\|g\|_\infty \leqslant 1} |\mathbb{E}(M_G(f) g)|$

$$= \sup_{\|g\|_\infty \leq 1} |\langle M_G(f), g\rangle|$$

Now, using the basic property of expectations

$$\langle f, M_G(g)\rangle = \langle M_G(f), M_G(g)\rangle \quad \left(\begin{matrix}\text{put } M_G \text{ on } f \text{ "for} \\ \text{nothing"}\end{matrix}\right)$$

and $\langle M_G(f), g\rangle = \langle M_G(f), M_G(g)\rangle$ because

$$\langle M_G(f), g\rangle = \mathbb{E}(M_G(f)g) = \mathbb{E}(M_G(M_G(f)g))$$

$$\overset{*}{=} \mathbb{E}(M_G(f)M_G(g)) = \langle M_G(f), M_G(g)\rangle.$$

* This uses $M_G(M_G(f)g) = M_G(f)M_G(g)$. So we have

$$\langle f, M_G(g)\rangle = \langle M_G(f), g\rangle$$

Some of you will recognise this as M_G being a self adjoint operator on $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

So $\|M_G(f)\|_1 = \sup_{\|g\|_\infty \leq 1} |\langle M_G(f), g\rangle| = \sup_{\|g\|_\infty \leq 1} |\langle f, M_G(g)\rangle|$

But $\|M_G(g)\|_\infty \leq \|g\|_\infty$ and therefore

$$\|M_G(f)\|_1 = \sup_{\|g\|_\infty \leq 1} |\langle f, M_G(g)\rangle| \leq \sup_{\|g\|_\infty \leq 1} |\langle f, g\rangle| = \|f\|_1$$

$\square$

<u>Definition</u>: Let $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $(f_n) \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ and G a sub-σ-field of $\mathcal{F}$. If (f_n) converges in $\|\cdot\|_1$ to f then $(M_G(f_n))$ is a Cauchy sequence in $L^1(\Omega, \mathcal{F}, \mathbb{P})$

We define $M_G'(f) = \lim_n M_G(f_n)$.

<u>Theorem</u>

(i) M_G' is a contractive projection of $L^1(\Omega, \mathcal{F}, \mathbb{P})$ onto $L^1(\Omega, G, \mathbb{P})$

(ii) M_G' agrees with M_G on $L^2(\Omega, \mathcal{F}, \mathbb{P})$

(iii) M_G' is well defined.

(iv) If $f \geq 0$ then $M_G(f) \geq 0$

(v) $\mathbb{E}(M_G'(f)) = \mathbb{E}(f)$

(vi) If $f, g \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ then $M_G'(f M_G'(g)) = M_G'(f) M_G'(g)$

<u>Pf</u> (iii) If $(f_n), (g_n) \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $f_n \to f$ in $\|\cdot\|_1$ and $g_n \to f$ in $\| \ \|_1$ then $\|M_G(f_n) - M_G(g_n)\|_1 \leq \|f_n - g_n\|_1 \leq \|f_n - f\|_1 + \|f - g_n\|_1 \longrightarrow 0$ as $n \to \infty$. So $M_G^1(f) = \lim_n M_G(f_n) = \lim_n M_G(g_n)$ (we already know that they will converge).

(i) Suppose $(f_n) \subset L^2$ and $f_n \to f \in L^1$, then $M_G(f_n) \to M_G'(f)$ in $\|\cdot\|_1$ and so $\|M_G(f_n)\|_1 \to \|M_G'(f)\|_1$. But $\|M_G(f_n)\|_1 \leq \|f_n\|_1$ and $\|f_n\|_1 \to \|f\|_1$. So $\|M_G'(f)\|_1 \leq \|f\|_1$. So M_G' is contractive.
Observe that $(M_G(f_n)) \subset L^2$ and $M_G(f_n) \to M_G'(f)$ in $\|\cdot\|_1$. So $M_G'(M_G'(f)) = \lim_n M_G(M_G(f_n)) = \lim_n M_G(f_n) = M_G'(f)$.

(ii) If $f \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ then the sequence $f, f, f, \ldots$ converges to f in $\|\cdot\|_1$ so $M_G^1(f) = \lim_n M_G(f) = M_G(f)$.

iv) If $f \geqslant 0$ and $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ then — as in the preamble $f_n = f I_{\{f \leqslant n\}}$ are increasing and converge to f in L^1 norm. So $M_q'(f) = \lim_n M_q(f_n)$. But M_q is positivity preserving & hence order preserving, so $0 \leqslant M_q(f_1) \leqslant M_q(f_2) \leqslant \ldots \leqslant \sup_n M_q(f)$

$= M_q'(f)$.

(v) On L^2, $\mathbb{E}(M_q(f)) = \mathbb{E}(f)$, so if $f_n \to f \in L^1$ then $|\mathbb{E}(f_n) - \mathbb{E}(f)| = |\mathbb{E}(f_n - f)| \leqslant \mathbb{E}(|f_n - f|)$

$= \|f_n - f\|_1 \longrightarrow 0$ as $n \to \infty$.

So $\mathbb{E}(f_n) \to \mathbb{E}(f)$. But $\mathbb{E}(M_q(f_n)) = \mathbb{E}(f_n)$ and by the above $\mathbb{E}(M_q(f_n)) \to \mathbb{E}(M_q'(f))$, so $\mathbb{E}(M_q'(f)) = \mathbb{E}(f)$.

(vi) Since $g \in L^2$, $M_q'(g) = M_q(g)$ and f is the $\|\cdot\|_1$ limit of a sequence of bounded functions, as in the preamble. However, for this problem we are going to "modify" our sequence of bounded functions (f_n) to ensure that they converge to f in $\|\cdot\|_2$ — this will imply that they converge in L^1, to f, but more than this it will imply that $f_n M_q(g)$ converges to $f M_q(g)$ in $\|\cdot\|_1$. Suppose first of all that $f \in L^2$ and $f \geqslant 0$. Let $f_n = f I_{\{f \leqslant n\}}$ as before note that $f f_n = f_n^2$ and that $f_n^2 \uparrow f^2$ and so – Monotone convergence – $\lim_n \int_\Omega f_n^2 = \int_\Omega f^2 \, d\mathbb{P}$. Now $\|f - f_n\|_2^2 = \int_\Omega (f - f_n)^2 d\mathbb{P} = \int_\Omega (f^2 - 2f_n f + f_n^2) d\mathbb{P} = \int_\Omega (f^2 - f_n^2) d\mathbb{P}$

and $\int_\Omega f^2 d\mathbb{P} = \lim_n \int_\Omega f_n^2 d\mathbb{P}$. So $f_n \to f$ in L^2 norm

It follows that $f_n M_G(g) \to f M_G(g)$ in $\|\cdot\|_1$,

because,

$$\| f_n M_G(g) - f M_G(g) \|_1 = \| (f_n - f) M_G(g) \|_1 \leq \|f_n - f\|_2 \cdot \|M_G(g)\|_2$$

and $\|f_n - f\|_2 \to 0$. By definition then,

$$M_G' (f M_G(g)) = \lim_n M_G (f_n M_G(g))$$

$$= \lim_n M_G(f_n) M_G(g)$$

But $M_G(f_n) \to M_G(f)$ in $\|\cdot\|_2$ because M_G is $\|\cdot\|_2$ continuous (contractive) and by a calculation similar to that above; $M_G(f_n) M_G(g) \to M_G(f) M_G(g)$ in L^1. So

$$M_G' (f M_G(g)) = M_G(f) M_G(g)$$

(†) Not really!